Dynamical Yang-Baxter equations, quasi-Poisson homogeneous spaces, and quantization

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1 Introduction

This paper is a continuation of [16]. Let us recall the main result of [16]. Let G be a Lie group, $\mathfrak{g} = \operatorname{Lie} G$, $U \subset G$ a connected closed Lie subgroup such that the corresponding subalgebra $\mathfrak{u} \subset \mathfrak{g}$ is reductive in $\mathfrak{g}$ (i.e., there exists an $\mathfrak{u}$-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{m}$), and $\Omega \in (\mathfrak{u} \otimes \mathfrak{u}) \oplus (\mathfrak{m} \otimes \mathfrak{m})$ a symmetric tensor. Take a solution $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ of the classical Yang-Baxter equation such that $\rho + \rho^{21} = \Omega$ and consider the corresponding Poisson Lie group structure π_ρ on G. Assuming additionally that

$$\rho + s \in \frac{\Omega}{2} + \left(\bigwedge\nolimits^2 \mathfrak{m}\right)^{\mathfrak{u}} \tag{1}$$

for some element $s \in \bigwedge^2 \mathfrak{g}$ that satisfies a certain "twist" equation, we establish a 1-1 correspondence between the moduli space of classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ with the symmetric part $\frac{\Omega}{2}$ and the set of all structures of Poisson homogeneous (G, π_ρ)-space on G/U. We emphasize that the first example of such a correspondence was found by Lu in [19].

We develop the results of [16] principally in two directions. First, we generalize the main result of [16]. We replace Poisson Lie groups (resp. Poisson homogeneous spaces) by quasi-Poisson Lie groups (resp. quasi-Poisson homogeneous spaces), but even in the Poisson case our result (see Theorem 8) is stronger than in [16]: condition (1) is relaxed now. We hope that now we present this result in its natural generality.

Secondly, we propose a partial quantization of the results of [16]. We explain how, starting from dynamical twist for a pair $(U\mathfrak{g}, \mathfrak{h})$ (where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ is its abelian subalgebra, and $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$), one can get a G-equivariant star-product on G/H (where $H \subset G$ are connected Lie groups corresponding to $\mathfrak{h} \subset \mathfrak{g}$). In the case $\mathfrak{g}$ is complex simple and $\mathfrak{h}$ is its Cartan subalgebra we give a representation-theoretic explanation of our results in terms of Verma modules. We also provide an analogue of these results for quantum universal enveloping algebras. Notice that results in this direction were obtained in the recent papers [7] and [3]. However, our approach is completely elementary (cf. [7]), and we emphasize the connection between

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star-products and dynamical twists very explicitly (cf. [3]). We also propose a method that allows one (under certain conditions) to obtain non-dynamical twists from dynamical ones. This is a quantization of the "classical" result obtained in [16, Appendix B].

Let us explain the structure of this paper in more details. Section 2 is devoted to the quasi-classical picture. In Subsection 2.1 we remind the definitions of classical dynamical r-matrices, quasi-Poisson Lie groups and their quasi-Poisson homogeneous spaces, and then formulate and prove the main result of this section, Theorem 8. In Subsection 2.2 we consider an example: the case of quasi-triangular (in the strict sense) classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{u}$ is its regular reductive subalgebra. Section 3 contains the construction of star-products from dynamical twists. As an example, we write down an explicit formula for an equivariant star-product on a regular semisimple coadjoint orbit of $SL(2)$ (note that this formula was also obtained as an example in [3]; similar formulas appeared earlier in physical papers) and observe its relation to certain Verma modules. In Section 4 we give a more conceptual explanation of the connection between equivariant quantization and Verma modules; our approach differs from that of [3]. In Section 5 we obtain an analogue of the results of Section 4 for the case of quantum universal enveloping algebras, therefore providing some examples of "quantum homogeneous spaces" related to dynamical twists for QUE-algebras. Finally, in Section 6 we present a way from dynamical twists to "usual" (non-dynamical) ones; we apply this construction to an explicit calculation of the universal twist for the universal enveloping algebra of two-dimensional nonabelian Lie algebra.

All Lie algebras in this paper assumed to be finite dimensional, and the ground field (except of Section 5) is $\mathbb{C}$.

When the paper was finished we got sad news about unexpected passing away of Joseph Donin. We dedicate this paper to his memory.

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2 Quasi-Poisson homogeneous spaces and classical dynamical r-matrices

2.1 General results

In this section we describe a connection between quasi-Poisson homogeneous spaces and classical dynamical r-matrices (see Theorem 8).

First we recall some definitions. Suppose G is a Lie group, $U \subset G$ its connected Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{u}$ be the corresponding Lie algebras.

Choose a basis $x_1, \ldots, x_r$ in $\mathfrak{u}$. Denote by D the formal neighborhood of zero in $\mathfrak{u}^*$. By functions from D to a vector space V we mean elements of the space $V[[x_1, \ldots, x_r]]$, where x_i are regarded as coordinates on D. Further, if $\omega \in \Omega^k(D, V)$ is a k-form on D with values in vector space V, then by $\overline{\omega} : D \to \bigwedge^k \mathfrak{u} \otimes V$ we denote the corresponding function.

Definition 1 (see [10]). *Classical dynamical r-matrix for the pair* $(\mathfrak{g}, \mathfrak{u})$ is an $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the *classical dynamical Yang-Baxter equation (CDYBE)*:

$$\mathrm{Alt}(\overline{dr}) + \mathrm{CYB}(r) = 0,$$

where $\mathrm{CYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$, and for $x \in \mathfrak{g}^{\otimes 3}$ we set $\mathrm{Alt}(x) = x^{123} + x^{231} + x^{312}$.

We will also require the *quasi-unitarity property*:

$$r + r^{21} = \Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}.$$

It is easy to see that if r satisfies the CDYBE and the quasi-unitarity condition, then Ω is constant.

We denote the set of all classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ such that $r + r^{21} = \Omega$ by $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Denote by $\mathbf{Map}(D, G)^{\mathfrak{u}}$ the set of all $\mathfrak{u}$-equivariant maps from D to G. Suppose that $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ is an $\mathfrak{u}$-equivariant function. Then for any $g \in \mathbf{Map}(D, G)^{\mathfrak{u}}$ define a function $r^g : D \to \mathfrak{g} \otimes \mathfrak{g}$ by

$$r^g = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(r - \overline{\eta_g} + \overline{\eta_g}^{21} + \tau_g),$$

where $\eta_g = g^{-1} dg$, and $\tau_g(\lambda) = (\lambda \otimes 1 \otimes 1)([\overline{\eta_g}^{12}, \overline{\eta_g}^{13}](\lambda))$. Then r^g is a classical dynamical r-matrix if and only if r is. The transformation $r \mapsto r^g$ is called a *gauge transformation*. In fact, it is an action of the group $\mathbf{Map}(D, G)^{\mathfrak{u}}$ on $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Following [10], we denote the moduli space $\mathbf{Map}_0(D, G)^{\mathfrak{u}} \backslash \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ by $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ (here $\mathbf{Map}_0(D, G)^{\mathfrak{u}} = \{g \in \mathbf{Map}(D, G)^{\mathfrak{u}} : g(0) = e\}$).

Now we recall the definition of quasi-Poisson Lie groups and their quasi-Poisson homogeneous spaces (for details see [18, 1, 2]).

Definition 2. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, π_G a bivector field on G, and $\varphi \in \bigwedge^3 \mathfrak{g}$. A triple (G, π_G, φ) is called a *quasi-Poisson Lie group* if

$$\pi_G(gg') = (l_g)_* \pi_G(g') + (r_{g'})_* \pi_G(g),$$
$$\frac{1}{2}[\pi_G, \pi_G] = \overleftarrow{\varphi} - \overrightarrow{\varphi},$$
$$[\pi_G, \overleftarrow{\varphi}] = 0,$$

where l_g (resp. r_g) is left (resp. right) multiplication by g, $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is the left (resp. right) invariant tensor field on G corresponding to a and $[.,.]$ is the Schouten bracket of multivector fields.

Definition 3. Suppose that (G, π_G, φ) is a quasi-Poisson group, X is a homogeneous G-space equipped with a bivector field π_X. Then (X, π_X) is called a *quasi-Poisson homogeneous* (G, π_G, φ)*-space* if

$$\pi_X(gx) = (l_g)_* \pi_X(x) + (\rho_x)_* \pi_G(g),$$
$$\frac{1}{2}[\pi_X, \pi_X] = \varphi_X$$

(here l_g denotes the mapping $x \mapsto g \cdot x$, ρ_x is the mapping $g \mapsto g \cdot x$, and φ_X is the trivector field on X induced by φ).

Now take $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\rho + \rho^{21} = \Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$. Let $\Lambda = \rho - \frac{\Omega}{2} \in \bigwedge^2 \mathfrak{g}$. Define a bivector field on G by $\pi_\rho = \overrightarrow{\rho} - \overleftarrow{\rho} = \overrightarrow{\Lambda} - \overleftarrow{\Lambda}$. Set $\varphi = \varphi_\rho = -\operatorname{CYB}(\rho)$. Then (G, π_ρ, φ) is a quasi-Poisson Lie group (such quasi-Poisson Lie groups are called *quasi-triangular*). Denote by $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$ the set of all (G, π_ρ, φ)-homogeneous quasi-Poisson structures on G/U. We will see that, under certain conditions, there is a bijection between $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ and $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$.

Assume that $b \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{u}}$ is such that $b + b^{21} = \Omega$. Let $B = b - \frac{\Omega}{2}$. Define a bivector field on G by $\tilde{\pi}_b^\rho = \overrightarrow{b} - \overleftarrow{\rho} = \overrightarrow{B} - \overleftarrow{\Lambda}$. Then there is a bivector field on G/U defined by $\pi_b^\rho(\underline{g}) = p_*(\tilde{\pi}_b^\rho(g))$ (here $p : G \to G/U$ is the canonical projection, and $\underline{g} = p(g)$). It is well defined, since b is $\mathfrak{u}$-invariant.

Proposition 1. *In this setting* $(G/U, \pi_b^\rho)$ *is a* (G, π_ρ, φ)*-quasi-Poisson homogeneous space iff* $\operatorname{CYB}(b) = 0$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$.

Proof. First we check the "multiplicativity" of π_b^ρ. For all $g \in G, u \in U$ we have

$$g \cdot \tilde{\pi}_b^\rho(u) + \pi_\rho(g) \cdot u = gu \cdot b - \rho \cdot gu = \tilde{\pi}_b^\rho(gu).$$

Using p_*, we get the required equality $\pi_b^\rho(\underline{g}) = g \cdot \pi_b^\rho(\underline{e}) + p_* \pi_\rho(g)$.

Now we need to prove that $\frac{1}{2}[\pi_b^\rho, \pi_b^\rho] = \varphi_{G/U}$ iff $\operatorname{CYB}(b) = 0$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. We check this directly:

$$\frac{1}{2}[\tilde{\pi}_b^\rho, \tilde{\pi}_b^\rho] = \frac{1}{2}\left([\overrightarrow{B}, \overrightarrow{B}] + [\overleftarrow{\Lambda}, \overleftarrow{\Lambda}]\right) = -\overrightarrow{\operatorname{CYB}(B)} + \overleftarrow{\operatorname{CYB}(\Lambda)} =$$
$$-\overrightarrow{\operatorname{CYB}(b)} + \overleftarrow{\varphi}.$$

Consequently, $\frac{1}{2}[\pi_b^\rho, \pi_b^\rho] = p_*(-\overrightarrow{\operatorname{CYB}(b)} + \overleftarrow{\varphi}) = -p_*(\overrightarrow{\operatorname{CYB}(b)}) + \varphi_{G/U}$. So we see that $\frac{1}{2}[\pi_b^\rho, \pi_b^\rho] = \varphi_{G/U}$ iff $\operatorname{CYB}(b) = 0$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. □

Suppose $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Proposition 2 (see [19]). $\operatorname{CYB}(r(0)) = 0$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. □

Corollary 3. *The correspondence* $r \mapsto \pi_{r(0)}^\rho$ *is a map from* $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$. □

Proposition 4 (see [16]). *If* $g \in \mathbf{Map}_0(D, G)^{\mathfrak{u}}$*, then* $\pi_{r(0)}^\rho = \pi_{r^g(0)}^\rho$. □

Corollary 5. *The correspondence $r \mapsto \pi^{\rho}_{r(0)}$ defines a map from $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$. $\square$

Now consider the following conditions:

$$\mathfrak{u} \text{ has an } \mathfrak{u}\text{-invariant complement } \mathfrak{m} \text{ in } \mathfrak{g}; \tag{2a}$$

$$\Omega \in (\mathfrak{u} \otimes \mathfrak{u}) \oplus (\mathfrak{m} \otimes \mathfrak{m}). \tag{2b}$$

Assume that (2a) holds and consider the algebraic variety

$$\mathcal{M}_\Omega = \left\{ x \in \frac{\Omega}{2} + \left(\bigwedge^2 \mathfrak{m}\right)^{\mathfrak{u}} \;\middle|\; \mathrm{CYB}(x) = 0 \text{ in } \bigwedge^3 (\mathfrak{g}/\mathfrak{u}) \right\}.$$

Theorem 6 (Etingof, Schiffman; see [10]). (1) *Any class $\mathcal{C} \in \mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ has a representative $r \in \mathcal{C}$ such that $r(0) \in \mathcal{M}_\Omega$. Moreover, this defines an embedding $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$.*

(2) *Assume that* (2b) *holds. Then the map $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$ defined above is a bijection.* $\square$

Proposition 7. *Assume that* (2a) *holds. Then the mapping $b \mapsto \pi^\rho_b$ from $\mathcal{M}_\Omega$ to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$ *is a bijection.*

Proof. Let's build the inverse mapping. Assume that π is a bivector field on G/U defining a structure of a (G, π_ρ, φ)-quasi-Poisson homogeneous space. Then $\pi(\underline{e}) \in \bigwedge^2 (\mathfrak{g}/\mathfrak{u}) = \bigwedge^2 \mathfrak{m}$. Consider $b = \frac{\Omega}{2} + \pi(\underline{e}) + p_*(\Lambda)$. We will prove that $b \in \mathcal{M}_\Omega$ and the mapping $\pi \mapsto b$ is inverse to the mapping $b \mapsto \pi^\rho_b$.

First we prove that $b \in (\bigwedge^2 \mathfrak{m})^{\mathfrak{u}} + \frac{\Omega}{2}$. For all $u \in U$ we have $\pi(\underline{e}) + p_*(\Lambda) = \pi(u \cdot \underline{e}) + p_*(\Lambda \cdot u) = u \cdot \pi(\underline{e}) + p_*(\pi_\rho(u)) + p_*(\Lambda \cdot u) = u \cdot \pi(\underline{e}) + p_*(u \cdot \rho - u \cdot \frac{\Omega}{2}) = u \cdot (\pi(\underline{e}) + p_*(\Lambda))$. This means that $\pi(\underline{e}) + p_*(\Lambda) \in (\bigwedge^2 \mathfrak{m})^{\mathfrak{u}}$.

Now we prove that $\pi = \pi^\rho_b$. By definition, $\pi^\rho_b(\underline{g}) = p_*(g \cdot \pi(\underline{e}) + g \cdot p_* \Lambda - \Lambda \cdot g) = \pi(\underline{g}) + p_*(g \cdot p_* \Lambda - \Lambda \cdot g - g \cdot \Lambda + \Lambda \cdot g) = \pi(\underline{g})$. So π^ρ_b defines a structure of (G, π_ρ, φ)-quasi-Poisson homogeneous space. By Proposition 1, this means that $b \in \mathcal{M}_\Omega$. $\square$

Theorem 8. *Suppose* (2a) *and* (2b) *are satisfied. Then the map $r \mapsto \pi^\rho_{r(0)}$ from $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$ *is a bijection.*

Proof. This theorem follows from Theorem 6 and Proposition 7. $\square$

Remark 1. If $\varphi = -\mathrm{CYB}(\rho) = 0$, then (G, π_ρ) is a Poisson Lie group. In this case we get a bijection between $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ and the set of all Poisson (G, π_ρ)-homogeneous structures on G/U.

Remark 2. Assume that only (2a) holds. Clearly, in this case the map $r \mapsto \pi^\rho_{r(0)}$ defines an embedding $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \hookrightarrow \mathbf{Homsp}(G, \pi_\rho, \varphi, U)$.

Remark 3. If (2a) fails, then the space $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ may be infinite-dimensional (see [20]), while $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$ is always finite-dimensional.

2.2 Example: the semisimple case

Assume that $\mathfrak{g}$ is a semisimple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\mathbf{R}$ the corresponding root system. Suppose $\langle .,. \rangle$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, and $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is the corresponding tensor. We will describe $\mathcal{M}_\Omega$ for a reductive Lie subalgebra $\mathfrak{u} \subset \mathfrak{g}$ containing $\mathfrak{h}$.

Precisely, consider a set $\mathbf{U} \subset \mathbf{R}$ such that $\mathfrak{u} = \mathfrak{h} \oplus \sum_{\alpha \in \mathbf{U}} \mathfrak{g}_\alpha$ is a reductive Lie subalgebra. In this case we will call $\mathbf{U}$ *reductive* (in other words, a set $\mathbf{U} \subset \mathbf{R}$ is reductive iff $(\mathbf{U}+\mathbf{U}) \cap \mathbf{R} \subset \mathbf{U}$ and $-\mathbf{U} = \mathbf{U}$). Note that in this situation condition (2a) is satisfied, since $\mathfrak{m} = \sum_{\alpha \in \mathbf{R} \backslash \mathbf{U}} \mathfrak{g}_\alpha$ is an $\mathfrak{u}$-invariant complement to $\mathfrak{u}$ in $\mathfrak{g}$.

Fix $E_\alpha \in \mathfrak{g}_\alpha$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$ for all $\alpha \in \mathbf{R}$. Then $\Omega = \Omega_{\mathfrak{h}} + \sum_{\alpha \in \mathbf{R}} E_\alpha \otimes E_{-\alpha}$, where $\Omega_{\mathfrak{h}} \in S^2\mathfrak{h}$. Notice that (2b) is also satisfied.

Proposition 9. *Suppose that $x = \sum_{\alpha \in \mathbf{R}} x_\alpha E_\alpha \otimes E_{-\alpha}$. Then $x + \frac{\Omega}{2} \in \mathcal{M}_\Omega$ iff*

$$x_\alpha = 0 \text{ for } \alpha \in \mathbf{U}; \tag{3a}$$

$$x_{-\alpha} = -x_\alpha \text{ for } \alpha \in \mathbf{R}; \tag{3b}$$

$$\text{if } \alpha, \beta \in \mathbf{R}\backslash\mathbf{U}, \gamma \in \mathbf{U}, \alpha+\beta+\gamma = 0, \text{ then } x_\alpha + x_\beta = 0; \tag{3c}$$

$$\text{if } \alpha, \beta, \gamma \in \mathbf{R}\backslash\mathbf{U}, \alpha+\beta+\gamma = 0, \text{ then } x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha = -1/4. \tag{3d}$$

Note that (3c) is equivalent to the following condition:

$$\text{if } \alpha \in \mathbf{R}\backslash\mathbf{U}, \beta \in \mathbf{U}, \text{then } x_{\alpha+\beta} = x_\alpha.$$

Proof. It is easy to see that $x \in (\bigwedge^2 \mathfrak{m})^{\mathfrak{h}}$ iff (3a) and (3b) are satisfied.

Suppose that $c_{\alpha\beta}$ are defined by $[E_\alpha, E_\beta] = c_{\alpha\beta} E_{\alpha+\beta}$.

For any $\gamma \in \mathbf{U}$ we have

$$[E_\gamma, x] = \sum_{\alpha \in \mathbf{R}\backslash\mathbf{U}} x_\alpha([E_\gamma, E_\alpha] \otimes E_{-\alpha} + E_\alpha \otimes [E_\gamma, E_{-\alpha}]) =$$

$$\sum_{\alpha,\beta \in \mathbf{R}\backslash\mathbf{U}, \alpha+\beta+\gamma=0} (x_\alpha c_{\gamma\alpha} E_{-\beta} \otimes E_{-\alpha} - x_\alpha c_{\gamma\alpha} E_{-\alpha} \otimes E_{-\beta}) =$$

$$\sum_{\alpha,\beta \in \mathbf{R}\backslash\mathbf{U}, \alpha+\beta+\gamma=0} (x_\beta c_{\gamma\alpha} - x_\alpha c_{\gamma\beta})\, E_{-\alpha} \otimes E_{-\beta} =$$

$$\sum_{\alpha,\beta \in \mathbf{R}\backslash\mathbf{U}, \alpha+\beta+\gamma=0} (x_\alpha + x_\beta) c_{\gamma\alpha} E_{-\alpha} \otimes E_{-\beta}.$$

Thus x is $\mathfrak{u}$-invariant if and only if $x_\alpha + x_\beta = 0$ for all $\alpha, \beta \in \mathbf{R}\backslash\mathbf{U}$ such that $\alpha + \beta \in \mathbf{U}$.

Finally, we calculate $\mathrm{CYB}\left(x + \frac{\Omega}{2}\right) = \mathrm{CYB}(x) + \mathrm{CYB}\left(\frac{\Omega}{2}\right)$ (see [1]):

$$
\begin{aligned}
&\mathrm{CYB}(x) = \\
&\sum_{\alpha,\beta\in\mathbf{R}} x_\alpha x_\beta\big([E_\alpha, E_\beta]\otimes E_{-\alpha}\otimes E_{-\beta} + E_\alpha\otimes[E_{-\alpha},E_\beta]\otimes E_{-\beta} + \\
&E_\alpha\otimes E_\beta\otimes[E_{-\alpha},E_{-\beta}]\big) = \\
&\sum_{\alpha,\beta,\gamma\in\mathbf{R},\alpha+\beta+\gamma=0} \big(x_\alpha x_\beta c_{\alpha\beta} E_{-\gamma}\otimes E_{-\alpha}\otimes E_{-\beta} - \\
&x_\alpha x_\beta c_{\alpha\beta} E_{-\alpha}\otimes E_{-\gamma}\otimes E_{-\beta} + x_\alpha x_\beta c_{\alpha\beta} E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma}\big) = \\
&\sum_{\alpha,\beta,\gamma\in\mathbf{R},\alpha+\beta+\gamma=0} c_{\alpha\beta}(x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma) E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma},
\end{aligned}
$$

$$
\mathrm{CYB}\left(\frac{\Omega}{2}\right) \equiv \frac{1}{4}\sum_{\alpha,\beta,\gamma\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} c_{\alpha\beta} E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma}
$$
$$
(\operatorname{mod} \mathfrak{u}\otimes\mathfrak{g}\otimes\mathfrak{g} + \mathfrak{g}\otimes\mathfrak{u}\otimes\mathfrak{g} + \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{u}).
$$

So the image of $\mathrm{CYB}\left(x+\frac{\Omega}{2}\right)$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$ vanishes if and only if the condition (3d) is satisfied. $\square$

Proposition 10. *Suppose $\Pi\subset\mathbf{R}$ is a set of simple roots, $\mathbf{R}_+$ is the corresponding set of positive roots. Choose a subset $\Delta\subset\Pi$ such that $\mathbf{N} = (\operatorname{span}\Delta)\cap\mathbf{R}$ contains $\mathbf{U}$. Find $h\in\mathfrak{h}$ such that $\alpha(h)\notin\pi i\mathbb{Z}$ for $\alpha\in\mathbf{N}\setminus\mathbf{U}$ and $\alpha(h)\in\pi i\mathbb{Z}$ for $\alpha\in\mathbf{U}$. Then x_α defined by*

$$
x_\alpha = \begin{cases} 0, & \alpha\in\mathbf{U} \\ \frac{1}{2}\coth\alpha(h), & \alpha\in\mathbf{N}\setminus\mathbf{U} \\ \pm 1/2, & \alpha\in\pm\mathbf{R}_+\setminus\mathbf{N} \end{cases}
$$

satisfies (3a)–(3d). *Moreover, any function satisfying* (3a)–(3d) *is of this form.*

First we prove the second part of the proposition. Set

$$
\mathbf{P} = \{\alpha \mid x_\alpha \neq -1/2\}.
$$

It is obvious that $\mathbf{U}\subset\mathbf{P}$.

Lemma 11. $\mathbf{P}$ *is parabolic.*

Proof. Obviously, $\mathbf{P}\cup(-\mathbf{P}) = \mathbf{R}$.

We have to prove that if $\alpha,\beta\in\mathbf{P}$ and $\alpha+\beta\in\mathbf{R}$, then $\alpha+\beta\in\mathbf{P}$. We do it by considering several cases. If $\alpha,\beta\in\mathbf{U}$, then $\alpha+\beta\in\mathbf{U}\subset\mathbf{P}$. If $\alpha\in\mathbf{P}\setminus\mathbf{U}$ and $\beta\in\mathbf{U}$, then $x_{\alpha+\beta} = x_\alpha \neq -1/2$ by (3c) and $\alpha+\beta\in\mathbf{P}$. If $\alpha,\beta\in\mathbf{P}\setminus\mathbf{U}$, there are two possibilities. If $\alpha+\beta\in\mathbf{U}$, then there is nothing to prove. If $\alpha+\beta\notin\mathbf{U}$, then, by (3d), $x_\alpha x_\beta - x_{\alpha+\beta}(x_\alpha+x_\beta) = -1/4$. If $x_{\alpha+\beta} = -1/2$, then from this equation it follows that $x_\alpha = -1/2$. Consequently, $\alpha+\beta\in\mathbf{P}$. $\square$

Since $\mathbf{P}$ is parabolic, there exists a set of positive roots $\Pi \subset \mathbf{R}$ and a subset $\Delta \subset \Pi$ such that $\mathbf{P} = \mathbf{R}_+ \cup \mathbf{N}$ (see [6], chapter VI, § 1, proposition 20); here $\mathbf{R}_+$ is the set of positive roots corresponding to Π, and $\mathbf{N} = (\operatorname{span} \Delta) \cap \mathbf{R}$ is the Levi subset corresponding to Δ.

Let $\mathbf{N}_+ = \mathbf{N} \cap \mathbf{R}_+$ be the set of positive roots in $\mathbf{N}$ corresponding to Δ. For all $\alpha \in \Delta \backslash \mathbf{U}$ let $y_\alpha = \operatorname{arccoth} 2x_\alpha$, for $\alpha \in \Delta \cap \mathbf{U}$ let $y_\alpha = 0$. Find $h \in \mathfrak{h}$ such that $y_\alpha = \alpha(h)$. Now we prove that h satisfies Proposition 10.

Lemma 12. *$\alpha(h) \notin \pi i \mathbb{Z}$ and $x_\alpha = \frac{1}{2} \coth \alpha(h)$ for all $\alpha \in \mathbf{N} \backslash \mathbf{U}$; $\alpha(h) \in \pi i \mathbb{Z}$ for $\alpha \in \mathbf{U}$.*

Proof. It is enough to prove this for α positive, so that we can use the induction on the length $l(\alpha)$. The case $l(\alpha) = 1$ is trivial. Suppose that $l(\alpha) = k$. Then we can find $\alpha' \in \mathbf{N}_+$ and $\alpha_k \in \Delta$ such that $l(\alpha') = k - 1$ and $\alpha = \alpha' + \alpha_k$. Consider two cases.

First, suppose that $\alpha \in \mathbf{U}$.

If $\alpha_k \in \mathbf{U}$, then $\alpha' \in \mathbf{U}$. By induction, $\alpha(h) = \alpha'(h) \in \pi i \mathbb{Z}$.

If $\alpha_k \notin \mathbf{U}$, then $\alpha' \notin \mathbf{U}$. By induction assumption, $x_{\alpha'} = \frac{1}{2} \coth \alpha'(h)$. From (3c) it follows that $0 = x_{\alpha'} + x_{\alpha_k} = \frac{1}{2}(\coth \alpha'(h) + \coth \alpha_k(h))$ and, consequently, $\alpha(h) \in \pi i \mathbb{Z}$.

Now suppose that $\alpha \notin \mathbf{U}$.

If $\alpha_k \in \mathbf{U}$, then $\alpha' \notin \mathbf{U}$. Since $\alpha_k(h) = 0$, by (3c) we have $x_\alpha = x_{\alpha' + \alpha_k} = x_{\alpha'} = \frac{1}{2} \coth \alpha'(h) = \frac{1}{2} \coth \alpha(h)$.

When $\alpha_k \notin \mathbf{U}$, then there are two possibilities again. If $\alpha' \in \mathbf{U}$, then by induction $\alpha'(h) \in \pi i \mathbb{Z}$. By (3c), $0 = x_\alpha + x_{-\alpha_k}$. Consequently, $x_\alpha = x_{\alpha_k} = \frac{1}{2} \coth \alpha_k(h) = \frac{1}{2} \coth \alpha(h)$. If $\alpha' \notin \mathbf{U}$, then, by (3d), $x_\alpha x_{-\alpha'} + x_{-\alpha'} x_{-\alpha_k} + x_{-\alpha_k} x_\alpha = -1/4$. This equation can be rewritten as

$$x_\alpha = \frac{1/4 + x_{\alpha'} x_{\alpha_k}}{x_{\alpha'} + x_{\alpha_k}} = \frac{1}{2} \cdot \frac{1 + \coth \alpha'(h) \coth \alpha_k(h)}{\coth \alpha'(h) + \coth \alpha_k(h)} = \frac{1}{2} \coth \alpha(h),$$

and the lemma is proved. □

To prove the first part of the proposition we need the following root theory lemma.

Lemma 13. *Suppose $\mathbf{P} \subset \mathbf{R}$ is parabolic. Then $\mathbf{Y} = \mathbf{R} \backslash \mathbf{P}$ has the following properties:*

$$(-\mathbf{Y}) \cap \mathbf{Y} = \emptyset; \tag{4a}$$

$$(\mathbf{Y} + \mathbf{Y}) \cap \mathbf{R} \subset \mathbf{Y}; \tag{4b}$$

$$\textit{if } \alpha \in \mathbf{Y}, \beta \in \mathbf{R} \backslash \mathbf{Y} \textit{ and } \alpha - \beta \in \mathbf{R}, \textit{ then } \alpha - \beta \in \mathbf{Y}. \tag{4c}$$

Proof. Since (4a) is obvious and (4b) follows from (4a) and (4c), we prove only the last property: if $\alpha \in \mathbf{Y}$ and $\beta \in \mathbf{P}$ are such that $\alpha - \beta \in \mathbf{P}$, then, since $\mathbf{P}$ is parabolic, we would have $\alpha = (\alpha - \beta) + \beta \in \mathbf{P}$. So $\alpha - \beta \in \mathbf{Y}$. □

Now we just check (3a)–(3d) directly. Suppose that $\mathbf{N}$ is defined as in the proposition. Let $\mathbf{Y} = \mathbf{R}_+ \backslash \mathbf{N}$. Then $\mathbf{P} = \mathbf{R} \backslash \mathbf{Y} = -\mathbf{R}_+ \cup \mathbf{N}$ is a parabolic set, and $\mathbf{Y}$ satisfies (4a)–(4c).

Lemma 14. *Suppose x_α is as defined in Proposition 10. Then x_α satisfies* (3a)–(3d).

Proof. (3a) we have already, (3b) is trivial.

To prove (3c), consider the following cases. First, take $\alpha, \beta \in \mathbf{N}\backslash\mathbf{U}$, $\gamma \in \mathbf{U}$, $\alpha+\beta+\gamma = 0$. Then $x_\alpha + x_\beta = \frac{1}{2}(\coth\alpha(h) + \coth\beta(h)) = 0$ as $\alpha+\beta = -\gamma \in \mathbf{U}$. The case $\alpha \in \mathbf{N}\backslash\mathbf{U}, \beta \in \mathbf{R}\backslash\mathbf{N}, \gamma \in \mathbf{U}$ is impossible, because then we would have $\beta = -\alpha - \gamma \in \mathbf{N}$. The case $\alpha, \beta \in \pm\mathbf{Y}, \gamma \in \mathbf{U}$ is also impossible, because $-\gamma = \alpha + \beta \in \pm\mathbf{Y}$. Finally, if $\alpha \in \pm\mathbf{Y}, \beta \in \mp\mathbf{Y}, \gamma \in \mathbf{U}$, then $x_\alpha + x_\beta = \pm\frac{1}{2} \mp \frac{1}{2} = 0$.

Condition (3d) can be proved in a similar way. □

Now to summarize:

Theorem 15. *Suppose $U \subset G$ is the connected Lie subgroup corresponding to $\mathfrak{u} \subset \mathfrak{g}$. Take $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\rho + \rho^{21} = \Omega$ and set $\varphi = -\operatorname{CYB}(\rho)$. Then any (G, π_ρ, φ)-homogeneous quasi-Poisson space structure on G/U is exactly of the form $\pi = \pi^\rho_{x+\Omega/2}$ for some $x = \sum_{\alpha\in R} x_\alpha E_\alpha \otimes E_{-\alpha}$, where x_α is defined in Proposition 10.* □

Remark 4. Let ρ be any solution of the classical Yang-Baxter equation such that $\rho + \rho^{21} = \Omega$ (see [5]). Then (G, π_ρ) is a Poisson Lie group and therefore Theorem 15 provides the list of all (G, π_ρ)-homogeneous Poisson space structures on G/U.

Remark 5. In [8], Drinfeld assigned to each point of any Poisson homogeneous space a Lagrangian (i.e., maximal isotropic) subalgebra in the corresponding double Lie algebra. Roughly speaking, this construction gives a one-to-one correspondence between Poisson homogeneous spaces up to isomorphism and Lagrangian subalgebras up to conjugation. In fact, literally the same is true in the quasi-Poisson case (see [15]). In our situation for any $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\rho + \rho^{21} = \Omega$ the Manin pair that corresponds to the (quasi-)Poisson Lie group $(G, \pi_\rho, \varphi_\rho)$ is the same and equals $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}_{\mathrm{diag}})$; here $\mathfrak{g} \times \mathfrak{g}$ is equipped with the invariant scalar product

$$Q((x, y), (x', y')) = \langle x, x' \rangle - \langle y, y' \rangle,$$

and $\mathfrak{g}_{\mathrm{diag}}$ is the image of the diagonal embedding $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. By [8, 15] we have a bijection between $(G, \pi_\rho, \varphi_\rho)$-homogeneous (quasi-)Poisson space structures on G/U and Lagrangian subalgebras $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{g}$ such that $\mathfrak{l} \cap \mathfrak{g}_{\mathrm{diag}} = \mathfrak{u}_{\mathrm{diag}}$ (this $\mathfrak{l}$ corresponds by Drinfeld to the base point $\underline{e} \in G/U$). Since all (quasi-)Poisson Lie groups $(G, \pi_\rho, \varphi_\rho)$ are related by twisting, we conclude that the Lagrangian subalgebra $\mathfrak{l}$ corresponding to $\pi^\rho_{x+\Omega/2}$ defined in Theorem 15 is independent of ρ (see [15, 16]). Using Drinfeld's definition, it is easy to compute $\mathfrak{l}$ by taking, for example, $\rho = \frac{\Omega}{2}$:

Proposition 16. *Under the notation of Proposition 10 and Theorem 15 let*

$$\mathfrak{b}_\pm = \mathfrak{h} \oplus \sum_{\alpha\in\mathbf{R}_+} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{n} = \mathfrak{h} \oplus \sum_{\alpha\in\mathbf{N}} \mathfrak{g}_\alpha,$$

$\mathfrak{p}_\pm = \mathfrak{n} + \mathfrak{b}_\pm$, *and* $\theta = \exp(2\,\mathrm{ad}_h) \in \operatorname{Aut}\mathfrak{n}$. *Then*

$$\mathfrak{l} = \{(x,y) \in \mathfrak{p}_- \times \mathfrak{p}_+ \,|\, \theta(p_-(x)) = p_+(y)\},$$

where $p_\pm : \mathfrak{p}_\pm \to \mathfrak{n}$ *are the canonical projections.* $\square$

3 Dynamical twists and equivariant quantization

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ its abelian subalgebra. We will use the standard Hopf algebra structure on the universal enveloping algebra $U\mathfrak{g}$ (and denote by Δ the comultiplication, by ε the counit, etc.). Suppose a meromorphic function

$$J : \mathfrak{h}^* \to (U\mathfrak{g} \otimes U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$$

is a *quantum dynamical twist*, i.e.,

$$J(\lambda)^{12,3} J(\lambda - \hbar h^{(3)})^{12} = J(\lambda)^{1,23} J(\lambda)^{23} \tag{5}$$

and

$$(\varepsilon \otimes \mathrm{id})(J(\lambda)) = (\mathrm{id} \otimes \varepsilon)(J(\lambda)) = 1. \tag{6}$$

Here

$$J(\lambda - \hbar h^{(3)})^{12} =$$
$$J(\lambda) \otimes 1 - \hbar \sum_i \frac{\partial J}{\partial \lambda_i}(\lambda) \otimes h_i + \frac{\hbar^2}{2} \sum_{i,j} \frac{\partial^2 J}{\partial \lambda_i \partial \lambda_j}(\lambda) \otimes h_i h_j - \dots,$$

h_i form a basis in $\mathfrak{h}$, and λ_i are the corresponding coordinates on $\mathfrak{h}^*$. We also use the standard notation $A^{12,3} = (\Delta \otimes \mathrm{id})(A)$, $A^{23} = 1 \otimes A$, etc., for any $A \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\hbar]]$.

Let G be a connected Lie group that corresponds to $\mathfrak{g}$. Assume that there exists the closed connected subgroup $H \subset G$ corresponding to $\mathfrak{h}$. Identify $C^\infty(G/H)$ with right H-invariant smooth functions on G.

Fix any $\lambda \in \operatorname{Dom} J$ and for any $f_1, f_2 \in C^\infty(G/H)$ define $f_1 \star_\lambda f_2 = \overrightarrow{J(\lambda)}(f_1, f_2) := (m \circ \overrightarrow{J(\lambda)})(f_1 \otimes f_2)$, where $\overrightarrow{J(\lambda)}$ is the left-invariant differential operator corresponding to $J(\lambda) \in (U\mathfrak{g} \otimes U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$, and m is the usual multiplication in $C^\infty(G/H)$ (extended naturally on $C^\infty(G/H)[[\hbar]]$). Since $J(\lambda)$ is $\mathfrak{h}$-invariant, we have $f_1 \star_\lambda f_2 \in C^\infty(G/H)[[\hbar]]$. As usual, we extend $\star_\lambda$ on $C^\infty(G/H)[[\hbar]]$.

Theorem 17. *The correspondence* $(f_1, f_2) \mapsto f_1 \star_\lambda f_2$ *is a* G*-equivariant star-product on* G/H.

Proof. We have

$$(f_1 \star_\lambda f_2) \star_\lambda f_3 = \left(m \circ (m \otimes \mathrm{id}) \circ \overrightarrow{J(\lambda)^{12,3} J(\lambda)^{12}} \right) (f_1 \otimes f_2 \otimes f_3),$$
$$f_1 \star_\lambda (f_2 \star_\lambda f_3) = \left(m \circ (m \otimes \mathrm{id}) \circ \overrightarrow{J(\lambda)^{1,23} J(\lambda)^{23}} \right) (f_1 \otimes f_2 \otimes f_3).$$

Since

$$J(\lambda - \hbar h^{(3)})^{12} - J(\lambda)^{12} \in (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h})[[\hbar]],$$

we see that

$$J(\lambda)^{12,3} J(\lambda)^{12} \equiv J(\lambda)^{1,23} J(\lambda)^{23} \mod (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h})[[\hbar]],$$

and $\overrightarrow{J(\lambda)^{12,3}J(\lambda)^{12}} = \overrightarrow{J(\lambda)^{1,23}J(\lambda)^{23}}$. This proves the associativity of $\star_\lambda$. The conditions $f \star_\lambda 1 = 1 \star_\lambda f = f$ follows from (6). The G-equivariance of $\star_\lambda$ is obvious. $\square$

Example 1. Suppose $\mathfrak{g} = \mathfrak{sl}(2)$, $G = SL(2)$. Let x, y, h be the standard basis in $\mathfrak{g}$, and $\mathfrak{h} = \mathbb{C}h$.

Consider the ABRR quantum dynamical twist J for $(\mathfrak{g}, \mathfrak{h})$, i.e.,

$$J(\lambda) = 1 + \sum_{n \geq 1} J_n(\lambda), \tag{7}$$

where

$$\begin{aligned} &J_n(\lambda) = \\ &\frac{(-1)^n}{n!} \hbar^n y^n \otimes (\lambda + \hbar(n+1-h))^{-1} \dots (\lambda + \hbar(2n-h))^{-1} x^n = \\ &\frac{(-1)^n}{n!} \hbar^n y^n \otimes x^n (\lambda - \hbar h)^{-1} (\lambda - \hbar(h+1))^{-1} \dots (\lambda - \hbar(h+n-1))^{-1} \end{aligned}$$

(see [4, 9]). Here we identify $\lambda \in \mathfrak{h}^*$ with $\lambda(h) \in \mathbb{C}$. Notice that $J(\lambda)$ is defined for $\lambda \neq 0$.

Clearly, for any $f_1, f_2 \in C^\infty(G/H)$ we have

$$f_1 \star_\lambda f_2 = f_1 f_2 + \sum_{n \geq 1} \overrightarrow{J^{(n)}_{\lambda,\hbar}}(f_1, f_2), \tag{8}$$

where

$$J^{(n)}_{\lambda,\hbar} = \frac{(-1)^n}{n!} \frac{\hbar^n}{\lambda(\lambda - \hbar) \dots (\lambda - (n-1)\hbar)} y^n \otimes x^n. \tag{9}$$

Equip $\mathfrak{g}$ with the invariant scalar product defined by $\langle a, b \rangle = \operatorname{Tr} ab$. Let us identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\langle \cdot, \cdot \rangle$. One can also consider G/H as a (co)adjoint G-orbit O_λ of $\lambda \in \mathfrak{h}^* \subset \mathfrak{g}^*$ (or $\frac{\lambda}{2} h \in \mathfrak{g}$). Denote by f_a the restriction onto O_λ of the linear function on $\mathfrak{g}^*$ generated by $a \in \mathfrak{g}$ (i.e., in terms of G/H we have $f_a(g) = \frac{\lambda}{2}\langle gHg^{-1}, a \rangle$). It is clear that $\overrightarrow{x^n} f_a = \overrightarrow{y^n} f_a = 0$ for all $a \in \mathfrak{g}$ and $n \geq 2$. Therefore

$$f_a \star_\lambda f_b - f_b \star_\lambda f_a = \hbar \overrightarrow{u_\lambda}(f_a, f_b) = \hbar f_{[a,b]},$$

where

$$u_\lambda = \frac{1}{\lambda}(x \otimes y - y \otimes x).$$

In other words, the quasiclassical limit of $\star_\lambda$ is exactly O_λ equipped with the Kirillov-Kostant-Souriau bracket.

Let us now restrict ourselves to regular functions on O_λ. Note that for any two such functions f_1, f_2 the series (8) has only finitely many non-vanishing summands. This allows us to fix the "deformation parameter" in (8) and (9) (i.e., set formally $\hbar = 1$). Of course, this makes sense if $\lambda \notin \mathbb{Z}_+$. Denote by A_λ the obtained algebra, i.e., the algebra of regular functions on O_λ with the multiplication $\star_\lambda$ (and $\hbar = 1$).

It is not hard to check directly that for any $a, b \in \mathfrak{g}$ we have

$$f_a \star_\lambda f_b = \left(1 - \frac{1}{\lambda}\right) f_a f_b + \frac{1}{2} f_{[a,b]} + \frac{\lambda}{2}\langle a, b\rangle. \tag{10}$$

Iterating (10), we see that $\star_\lambda$ is compatible with the standard (i.e., by polynomial degree) filtration on A_λ. Since

$$f_a \star_\lambda f_b - f_b \star_\lambda f_a = f_{[a,b]},$$

we get an algebra homomorphism $F : U\mathfrak{g} \to A_\lambda$ defined by $a \mapsto f_a$ for all $a \in \mathfrak{g}$. Obviously, F is filtered with respect to the standard filtrations on $U\mathfrak{g}$ and A_λ.

Consider the Casimir element $c = xy + yx + \frac{1}{2}h^2 \in U\mathfrak{g}$. Let us calculate $F(c)$. We have

$$\begin{aligned}&f_x \star_\lambda f_y + f_y \star_\lambda f_x + \frac{1}{2} f_h \star_\lambda f_h = \\ &\left(1 - \frac{1}{\lambda}\right)\left(2 f_x f_y + \frac{1}{2} f_h^2\right) + \frac{3\lambda}{2} = \left(1 - \frac{1}{\lambda}\right)\frac{\lambda^2}{2} + \frac{3\lambda}{2} = \frac{\lambda(\lambda+2)}{2}.\end{aligned}$$

Therefore $c - \frac{\lambda(\lambda+2)}{2} \in \operatorname{Ker} F$, and F induces the (filtered) homomorphism $\widetilde{F} : U\mathfrak{g} \Big/ \left(c - \frac{\lambda(\lambda+2)}{2}\right) \to A_\lambda$.

Let us now pass to the corresponding gradings. Since $\operatorname{gr} \widetilde{F}$ is obviously surjective on each graded component, and the dimensions of the corresponding graded components of $U\mathfrak{g} \Big/ \left(c - \frac{\lambda(\lambda+2)}{2}\right)$ and A_λ are the same, we see that $\operatorname{gr} \widetilde{F}$ is an isomorphism. Thus $\widetilde{F}$ is also an isomorphism, i.e.,

$$A_\lambda \simeq U\mathfrak{g} \Big/ \left(c - \frac{\lambda(\lambda+2)}{2}\right).$$

Notice that for $\lambda \notin \mathbb{Z}_+$ the ideal $\left(c - \frac{\lambda(\lambda+2)}{2}\right)$ is exactly the kernel of the natural homomorphism $U\mathfrak{g} \to \operatorname{End} M(\lambda)$, where $M(\lambda)$ is the Verma module with highest weight λ. Therefore we get an embedding $A_\lambda \hookrightarrow \operatorname{End} M(\lambda)$.

4 Verma modules and equivariant quantization

In this section we give an explanation of the appearance of Verma modules in Example 1.

4.1 General construction

Let $F = \mathbb{C}[G]$ be the algebra of polynomial functions on a simple complex Lie group G, which is an algebra generated by matrix elements of finite dimensional representations. Set $\mathfrak{g} = \operatorname{Lie} G$. We equip F by a structure of $U\mathfrak{g}$-module algebra via $(a, f) \mapsto \overrightarrow{a} f$.

Let H be a Cartan subgroup of G, and $\mathfrak{h} = \operatorname{Lie} H$. We also define $\operatorname{Fun}(G/H) = F[0] = \{f \in L \mid \overrightarrow{h} f = 0 \text{ for any } h \in \mathfrak{h}\}$.

Let M be a $\mathfrak{g}$-module. On $\operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$ we introduce a natural structure of algebra in the following way. Let $\varphi, \psi \in \operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$. Set

$$\varphi * \psi = (\operatorname{id} \otimes m) \circ (\varphi \otimes \operatorname{id}) \circ \psi, \tag{11}$$

where m is the multiplication in F. We notice that $\varphi * \psi \in \operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$. It is not difficult to see that the multiplication $*$ is associative and $e(m) = m \otimes 1$ is the identity element.

Remark 6. Any linear map $f : M \to M \otimes F$ can be considered as a function $\widetilde{f} : G \to \operatorname{End} M$ in the following way: $\widetilde{f}(g)(v) = f(v)(g)$. We notice that $\widetilde{\varphi * \psi} = \widetilde{\varphi} \cdot \widetilde{\psi}$. Elements of $\operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$ can be distinguished by the following lemma.

Lemma 18. *Let $\varphi \in \operatorname{Hom}(M, M \otimes F)$. Then $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$ if and only if the corresponding function $\widetilde{\varphi}$ satisfies the first order differential equation $\overrightarrow{a}\widetilde{\varphi}(g) = [\widetilde{\varphi}(g), a_M]$.* □

Corollary 19. *There exists an embedding $\operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F) \hookrightarrow \operatorname{End} M$ given by the formula $\varphi \mapsto \widetilde{\varphi}(e)$.*

Proof. The fact that this map is a homomorphism follows from the remark above. Injectivity follows from Lemma 18. □

Let us identify $\operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$ with its image in $\operatorname{End} M$ when appropriate.

Proposition 20. *There exists a homomorphism $U\mathfrak{g} \to \operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F)$ given by the formula $x \mapsto \operatorname{Ad}_{g^{-1}}(x)_M$.*

Proof. We have to verify that $\operatorname{Ad}_{g^{-1}}(x)$ satisfies the differential equation from Lemma 18, which is straightforward. □

Remark 7. The composition

$$U\mathfrak{g} \to \operatorname{Hom}_{\mathfrak{g}}(M, M \otimes F) \to \operatorname{End} M$$

is the standard homomorphism $U\mathfrak{g} \to \operatorname{End} M,\ x \mapsto x_M$.

4.2 Verma modules

Now fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and set $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. Let $M(\lambda)$ be the Verma module with the highest weight $\lambda \in \mathfrak{h}^*$ and the highest weight vector $\mathbb{I}_\lambda$. We call λ *generic* if $\langle \lambda, \alpha \rangle \notin \mathbb{Z}$ for any root α of $\mathfrak{g}$. It is well known that in this case $M(\lambda)$ is irreducible.

For any $\mathfrak{g}$-module V we will denote by $V[\mu]$ its subspace of all weight vectors of weight $\mu \in \mathfrak{h}^*$.

Now let us construct a map $\mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F) \to F[0]$ for any λ. Choosing $\varphi \in \mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F)$ we consider $\varphi(\mathbb{I}_\lambda) \in M(\lambda) \otimes F$. Clearly, $\varphi(\mathbb{I}_\lambda) = \mathbb{I}_\lambda \otimes f_\varphi + \sum_{\mu<\lambda} v_\mu \otimes f_\mu$, where $v_\mu \in M(\lambda)[\mu]$. Obviously, $f_\varphi \in F[0]$. The correspondence $\varphi \mapsto f_\varphi$ is the required map.

Lemma 21. *If $M(\lambda)$ is irreducible, then this map is an isomorphism of vector spaces.*

Proof. Since F is a direct sum of finite dimensional $\mathfrak{g}$-modules, it is enough to prove that $\mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes V) \cong V[0]$ if $\dim V < \infty$. This is well known (see [9]). □

Lemma 21 provides a structure of associative algebra on $F[0]$ since the space $\mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F)$ has such a structure. We would like to describe this structure in more details. Let us recall the definition of the universal dynamical twist (see [9]).

Let $\varphi \in \mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes V)$ and $\psi \in \mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes W)$, where V, W are finite dimensional $\mathfrak{g}$-modules. We have the following picture:

$$
\begin{aligned}
&M(\lambda) \xrightarrow{\psi} M(\lambda) \otimes W \xrightarrow{\varphi\otimes \mathrm{id}} M(\lambda) \otimes V \otimes W,\\
&\psi(\mathbb{I}_\lambda) = \mathbb{I}_\lambda \otimes u_\psi + \sum_{\mu<\lambda} v_\mu \otimes u_{\mu,\psi},\\
&\varphi(\mathbb{I}_\lambda) = \mathbb{I}_\lambda \otimes u_\varphi + \sum_{\mu<\lambda} v_\mu \otimes u_{\mu,\varphi},\\
&(\varphi \otimes \mathrm{id})\psi(\mathbb{I}_\lambda) = \mathbb{I}_\lambda \otimes u_{(\varphi\otimes\mathrm{id})\psi} + \sum_{\mu<\lambda} v_\mu \otimes u_{\mu,(\varphi\otimes\mathrm{id})\psi}.
\end{aligned}
$$

It turns out that there exists a universal series $J(\lambda) \in U\mathfrak{g}\widehat{\otimes}U\mathfrak{g}$ such that

$$u_{(\varphi\otimes\mathrm{id})\psi} = J(\lambda)_{V\otimes W}(u_\varphi \otimes u_\psi).$$

It is known that $J(\lambda) \in 1 \otimes 1 + (\mathfrak{n}_- \cdot U\mathfrak{n}_-)\widehat{\otimes}(U\mathfrak{b}_+ \cdot \mathfrak{n}_+)$ and its coefficients are rational functions on λ. Moreover, $J(\lambda)$ is a dynamical twist.

Proposition 22. $f_{\varphi*\psi} = (m \circ \overrightarrow{J(\lambda)})(f_\varphi \otimes f_\psi)$.

Proof. Taking into account that F is a sum of finite dimensional modules and the construction of $\varphi * \psi$ we get the required result. □

Corollary 23. *The formula $f_1 \star_\lambda f_2 = (m \circ \overrightarrow{J(\lambda)})(f_1 \otimes f_2)$ defines an associative product on $F[0]$.* □

In the sequel we denote by $F[0]_\lambda$ the obtained algebra.

It is known that

$$J(\lambda/\hbar) = 1 \otimes 1 + \hbar j(\lambda) + O(\hbar^2)$$

and $r(\lambda) = j(\lambda) - j(\lambda)^{21}$ is the classical triangular dynamical r-matrix (see [9]). It was also noticed in [16] that $r(\lambda)$ defines a family of G-invariant Poisson structures on G/H, which we denote by $\{\cdot,\cdot\}_\lambda$. Any such a structure is in fact coming from the Kirillov-Kostant-Souriau bracket on the coadjoint orbit $O_\lambda \subset \mathfrak{g}^*$.

Corollary 24. *The multiplication $f_1 \star_{\lambda/\hbar} f_2$ is an equivariant deformation quantization of the Poisson homogeneous structure $\{\cdot,\cdot\}_\lambda$ on G/H (and hence a quantization of the Kirillov-Kostant-Souriau bracket on O_λ).* □

Now let us discuss the image of $U\mathfrak{g}$ in $F[0]_\lambda$.

For this we have to compute $u_{x,\lambda}$ in $\mathrm{Ad}_{g^{-1}}(x)\mathbb{I}_\lambda = \mathbb{I}_\lambda \otimes u_{x,\lambda}(g) + \ldots$. We have the decomposition $U\mathfrak{g} = U\mathfrak{h} \oplus (\mathfrak{n}_- \cdot U\mathfrak{g} + U\mathfrak{g} \cdot \mathfrak{n}_+)$, and for any $a \in U\mathfrak{g}$ we define $(a)_0 \in U\mathfrak{h}$ as the corresponding projection. It is clear that $u_{x,\lambda}(g) = (\mathrm{Ad}_{g^{-1}}(x))_0(\lambda)$ (we identify $U\mathfrak{h}$ with polynomial functions on $\mathfrak{h}^*$).

We get the following

Proposition 25. $u_{x,\lambda} \star_\lambda u_{y,\lambda} = u_{xy,\lambda}$. □

Proposition 26. *The homomorphism $U\mathfrak{g} \to F[0]_\lambda$ defined by $x \mapsto u_{x,\lambda}$ is surjective for generic λ.*

Proof. We have the maps

$$U\mathfrak{g} \to U\mathfrak{g} \otimes F \to U\mathfrak{h} \otimes F[0] \to F[0]$$

defined by $x \mapsto \mathrm{Ad}_{g^{-1}}(x) \mapsto (\mathrm{Ad}_{g^{-1}}(x))_0 \mapsto (\mathrm{Ad}_{g^{-1}}(x))_0(\lambda)$.

If we set $\deg F = 0$, then the first two maps are filtered with respect to the standard filtration of $U\mathfrak{g}$.

If we go to the corresponding graded spaces we get the maps

$$S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes F \to S(\mathfrak{h}) \otimes F[0] \to F[0],$$

$a \mapsto \mathrm{Ad}_{g^{-1}}(a) \mapsto (\mathrm{Ad}_{g^{-1}}(a))_0 \mapsto (\mathrm{Ad}_{g^{-1}}(a))_0(\lambda)$. In this case the surjectivity of the composition map $S(\mathfrak{g}) \to F[0]$ for generic λ is well known. It follows now that the map $U\mathfrak{g} \to F[0]$ is also surjective. □

Let again λ be generic. Since $\mathrm{Hom}_\mathfrak{g}(M(\lambda), M(\lambda) \otimes F)$ is isomorphic to $F[0]_\lambda$ as an algebra, we get the following algebra homomorphisms:

$$U\mathfrak{g} \to F[0]_\lambda \to \mathrm{End}\, M(\lambda),$$

and the composition is the standard map $U\mathfrak{g} \to \mathrm{End}\, M(\lambda)$.

Corollary 27. *For any generic λ the images of $U\mathfrak{g}$ and $F[0]_\lambda$ in* $\mathrm{End}\, M(\lambda)$ *coincide.* □

5 Verma modules and quantum homogeneous spaces

Now we are going to give an analogue of the results from Section 4 to the case of quantum universal enveloping algebras. It will be convenient to develop first some formalism for a general Hopf algebra (see Subsection 5.1) and then proceed to quantum universal enveloping algebras and corresponding Verma modules (see Subsection 5.2).

5.1 General construction

Let A be a Hopf algebra (over an arbitrary field $\Bbbk$). As usual, we will denote by Δ (resp. ε, S) the comultiplication (resp. counit, antipode) in A. We will systematically use the Sweedler notation for comultiplication, i.e., $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, etc.

Assume M is a (left) A-module. We call an element $m \in M$ *locally finite* if $\dim Am < \infty$. Denote by M_{fin} the subset of all locally finite elements in M. Clearly, M_{fin} is a submodule in M. Similarly, we can consider locally finite elements in a right A-module N. For convenience, we will use the notation $N^{\mathrm{r}}_{\mathrm{fin}}$ for the submodule of all locally finite elements in this case.

Recall that the left (resp. right) adjoint action of A on itself is defined by the formula $\mathrm{ad}_x a = \sum_{(x)} x_{(1)} a S(x_{(2)})$ (resp. $\mathrm{ad}^{\mathrm{r}}_x a = \sum_{(x)} S(x_{(1)}) a x_{(2)}$). We denote by A_{fin} (resp. $A^{\mathrm{r}}_{\mathrm{fin}}$) the corresponding submodules of locally finite elements. Since $\mathrm{ad}_x(ab) = \sum_{(x)} \mathrm{ad}_{x_{(1)}}(a)\, \mathrm{ad}_{x_{(2)}}(b)$, we see that A_{fin} is a (unital) subalgebra in A; the same holds for $A^{\mathrm{r}}_{\mathrm{fin}}$. If the antipode S is invertible, then S defines an isomorphism between A_{fin} and $A^{\mathrm{r}}_{\mathrm{fin}}$. We will assume that S is invertible.

Fix a Hopf subalgebra F of the Hopf algebra $A^\star$ dual to A. In the sequel we will use the left and right regular actions of A on F defined respectively by the formulas $(\overrightarrow{a} f)(x) = f(xa)$ and $(f \overleftarrow{a})(x) = f(ax)$.

Now let M be a (left) A-module. Equip F with the left regular A-action and consider the space $\mathrm{Hom}_A(M, M \otimes F)$. For any $\varphi, \psi \in \mathrm{Hom}_A(M, M \otimes F)$ define $\varphi * \psi$ by formula (11). It is straightforward to verify that $\varphi * \psi \in \mathrm{Hom}_A(M, M \otimes F)$, and this definition equips $\mathrm{Hom}_A(M, M \otimes F)$ with a unital associative algebra structure.

Consider the map $\Phi : \mathrm{Hom}_A(M, M \otimes F) \to \mathrm{End}\, M$, $\varphi \mapsto u_\varphi$, defined by $u_\varphi(m) = (\mathrm{id} \otimes \varepsilon)(\varphi(m))$; here $\varepsilon(f) = f(1)$ is the counit in F. In other words, if $\varphi(m) = \sum_i m_i \otimes f_i$, then $u_\varphi(m) = \sum_i f_i(1) m_i$. Using the fact that ε is an algebra homomorphism it is easy to show that Φ is an algebra homomorphism as well.

Lemma 28. *The map* Φ *embeds* $\mathrm{Hom}_A(M, M \otimes F)$ *into* $\mathrm{End}\, M$.

Proof. If $\varphi \in \mathrm{Hom}_A(M, M \otimes F)$, $\varphi(m) = \sum_i m_i \otimes f_i$, then

$$\varphi(am) = a\varphi(m) = \sum_i \sum_{(a)} a_{(1)} m_i \otimes \overrightarrow{a_{(2)}} f_i,$$

and

$$u_\varphi(am) = \sum_i \sum_{(a)} (\overrightarrow{a_{(2)}} f_i)(1) a_{(1)} m_i = \sum_{(a)} a_{(1)} \left(\sum_i f(a_{(2)}) m_i \right).$$

Assume now that $u_\varphi = 0$, i.e., $\sum_{(a)} a_{(1)} \left(\sum_i f(a_{(2)}) m_i \right) = 0$ for any $a \in A$ and $m \in M$. Then, in particular,

$$0 = \sum_{(a)} S(a_{(1)}) a_{(2)} \left(\sum_i f(a_{(3)}) m_i \right) =$$

$$\sum_{(a)} \varepsilon(a_{(1)}) \left(\sum_i f(a_{(2)}) m_i \right) = \sum_i f_i(a) m_i$$

for any $a \in A$ and $m \in M$. Obviously, this means that $\varphi = 0$. □

From now on we assume that F contains all matrix elements of the (left) adjoint action of A on A_{fin}. Since F is closed under the antipode $(Sf)(x) = f(S(x))$, we see that this assumption is equivalent to the fact that F contains all matrix elements of the right adjoint action of A on $A^{\mathrm{r}}_{\mathrm{fin}}$.

Let $a \in A^{\mathrm{r}}_{\mathrm{fin}}$, i.e., for any $x \in A$ we have $\mathrm{ad}^{\mathrm{r}}_x a = \sum_i f_i(x) a_i$, where $f_i \in A^\star$, $a_i \in A$. In fact, we see that $f_i \in F$ by the assumption above. Define a linear map $\varphi_a : M \to M \otimes F$ by the formula $\varphi_a(m) = \sum_i a_i m \otimes f_i$. Clearly, φ_a is well defined.

Lemma 29. *For any $a \in A^{\mathrm{r}}_{\mathrm{fin}}$ we have $\varphi_a \in \mathrm{Hom}_A(M, M \otimes F)$.*

Proof. Let $b \in A$. Notice that

$$\sum_{(b)} b_{(1)} \,\mathrm{ad}^{\mathrm{r}}_{b_{(2)}} y = \sum_{(b)} b_{(1)} S(b_{(2)}) y b_{(3)} = y \sum_{(b)} \varepsilon(b_{(1)}) b_{(2)} = yb$$

for any $y \in A$. Therefore for any $x \in A$ we have

$$\sum_i f_i(x) a_i b = (\mathrm{ad}^{\mathrm{r}}_x a) b = \sum_{(b)} b_{(1)} \,\mathrm{ad}^{\mathrm{r}}_{b_{(2)}} \mathrm{ad}^{\mathrm{r}}_x a = \sum_{(b)} b_{(1)} \,\mathrm{ad}^{\mathrm{r}}_{x b_{(2)}} a =$$

$$\sum_{(b)} b_{(1)} \left(\sum_i f_i(x b_{(2)}) a_i \right) = \sum_{(b)} \sum_i (\overrightarrow{b_{(2)}} f_i)(x) b_{(1)} a_i,$$

and

$$\varphi_a(bm) = \sum_i a_i b m \otimes f_i = \sum_{(b)} \sum_i b_{(1)} a_i m \otimes \overrightarrow{b_{(2)}} f_i = b \varphi_a(m).$$

□

Denote by $\Psi : A^{\mathrm{r}}_{\mathrm{fin}} \to \mathrm{Hom}_A(M, M \otimes F)$ the linear map constructed above (i.e., $\Psi : a \mapsto \varphi_a$).

Lemma 30. *The map Ψ is an algebra homomorphism.*

Proof. Let $a, b \in A^{\mathrm{r}}_{\mathrm{fin}}$, $x \in A$, $\mathrm{ad}^{\mathrm{r}}_x a = \sum_i f_i(x) a_i$, $\mathrm{ad}^{\mathrm{r}}_x b = \sum_j g_j(x) b_j$. Then

$$\mathrm{ad}^{\mathrm{r}}_x(ab) = \sum_{(x)} \mathrm{ad}_{x_{(1)}}(a)\, \mathrm{ad}_{x_{(2)}}(b) =$$

$$\sum_{i,j} \sum_{(x)} f_i(x_{(1)}) g_j(x_{(2)}) a_i b_j = \sum_{i,j} (f_i g_j)(x) a_i b_j.$$

Thus

$$\varphi_{ab}(m) = \sum_{i,j} a_i b_j m \otimes f_i g_j = (\varphi_a * \varphi_b)(m)$$

for any $m \in M$. ☐

Remark 8. It follows directly from the definitions that the composition $\Phi\Psi$ equals the restriction to $A^{\mathrm{r}}_{\mathrm{fin}}$ of the canonical homomorphism $A \to \operatorname{End} M$, $a \mapsto a_M$.

Now consider $A^{\mathrm{r}}_{\mathrm{fin}}$, $\operatorname{Hom}_A(M, M \otimes F)$ and $\operatorname{End} M$ as right A-modules: $A^{\mathrm{r}}_{\mathrm{fin}}$ via right adjoint action, $\operatorname{Hom}_A(M, M \otimes F)$ via right regular action on F (i.e., $(\varphi \cdot a)(m) = (\mathrm{id} \otimes \overleftarrow{a})(\varphi(m))$), and $\operatorname{End} M$ in a standard way (i.e., $u \cdot a = \sum_{(a)} S(a_{(1)})_M u a_{(2)}{}_M$). Note that $A^{\mathrm{r}}_{\mathrm{fin}}$, $\operatorname{Hom}_A(M, M \otimes F)$ and $\operatorname{End} M$ equipped with these structures are indeed right A-module algebras, i.e., the multiplication map is a module morphism, and the unit is invariant.

Lemma 31. *The maps Φ and Ψ are morphisms of right A-modules.*

Proof. Straightforward. ☐

Corollary 32. *We have the following morphisms of right A-module algebras:*

$$A^{\mathrm{r}}_{\mathrm{fin}} \xrightarrow{\Psi} \operatorname{Hom}_A(M, M \otimes F)^{\mathrm{r}}_{\mathrm{fin}} \xrightarrow{\Phi} (\operatorname{End} M)^{\mathrm{r}}_{\mathrm{fin}},$$

and $\Phi\Psi$ is the restriction of the canonical morphism $A \to \operatorname{End} M$. ☐

5.2 QUE algebra case

Now suppose $A = \check{U}_q\mathfrak{g}$, where $\mathfrak{g}$ is a complex simple Lie algebra (see [11, §3.2.10] or [13]). We consider A as an algebra over $\overline{\mathbb{C}(q)}$, the algebraic closure of the field $\mathbb{C}(q)$ of rational functions on the indeterminate q. It is known that the adjoint action of A is not locally finite. The subalgebra $A_{\mathrm{fin}} \subset A$ was studied in [12, 13] (see also [11]).

Let $F = \mathbb{C}[G]_q$ be the quantized algebra of regular functions on an algebraic group G corresponding to $\mathfrak{g}$ (see [11, 17]). We can consider F as a Hopf subalgebra in $A^\star$. Clearly, F satisfies the requirements of the previous subsection. Notice that F is a sum of finite dimensional admissible A-modules with respect to both left and right regular actions of A (see [17]).

Denote by $\mathfrak{h}^*_{\mathbb{Q}}$ the $\mathbb{Q}$-span of the weight lattice of $\mathfrak{g}$. Consider the Verma module $M(\lambda)$ for A with the highest weight $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ and the highest weight

vector $\mathbb{I}_\lambda$. As in the classical case, if λ is generic (i.e., $\langle\lambda,\alpha\rangle \notin \mathbb{Z}$ for any root α of $\mathfrak{g}$), then $M(\lambda)$ is irreducible.

For any left A-module V we will denote by $V[\mu]$ its subspace of all weight vectors of weight $\mu \in \mathfrak{h}^*_{\mathbb{Q}}$.

Now for any $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ we construct a linear map

$$\Theta_\lambda : \mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F) \to F[0]$$

(here F is considered as an A-module via left regular action). Take $\varphi \in \mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F)$ and consider $\varphi(\mathbb{I}_\lambda) \in M(\lambda)\otimes F$. Clearly, $\varphi(\mathbb{I}_\lambda) = \mathbb{I}_\lambda \otimes f_\varphi + \sum_{\mu<\lambda} v_\mu \otimes f_\mu$, where $v_\mu \in M(\lambda)[\mu]$. We see that $f_\varphi \in F[0]$. The correspondence $\varphi \mapsto f_\varphi$ is the map of concern. Notice that both $\mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F)$ and $F[0]$ are right A-modules (via right regular action of A on F), and Θ_λ is compatible with these structures.

By the same argument as in the classical case (see Lemma 21), we have the following

Lemma 33. *If $M(\lambda)$ is irreducible, then Θ_λ is an isomorphism (of right A-modules).* □

Now for any generic λ we can use Θ_λ to transfer to $F[0]$ the product $*$ on $\mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F)$, which was constructed in the previous subsection.

Arguing like in the classical case, we see that $f_{\varphi*\psi} = (m \circ \overrightarrow{J_q(\lambda)})(f_\varphi \otimes f_\psi)$ for any $\varphi, \psi \in \mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F)$. Here $J_q(\lambda)$ is a universal quantum dynamical twist for A (see [9]). Therefore we get

Corollary 34. *The formula $f_1 \star_\lambda f_2 = (m \circ \overrightarrow{J_q(\lambda)})(f_1 \otimes f_2)$ defines an associative product on $F[0]$, and $\Theta_\lambda : (\mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F), *) \to (F[0], \star_\lambda)$ is an algebra isomorphism.* □

Let us denote by $F[0]_\lambda$ the algebra $(F[0], \star_\lambda)$.

Remark 9. Obviously, $F[0]_\lambda$ is a right A-module algebra (i.e., $(f_1 \star_\lambda f_2)\overleftarrow{a} = \sum_{(a)} f_1\overleftarrow{a_{(1)}} \star_\lambda f_2\overleftarrow{a_{(2)}}$ and $1\overleftarrow{a} = \varepsilon(a)1$ for any $a \in A$).

Let us identify $\mathrm{Hom}_A(M(\lambda), M(\lambda)\otimes F)$ and $F[0]_\lambda$ via Θ_λ. Note that the right regular action on $F[0] \subset F$ is locally finite, i.e., $(F[0])^{\mathrm{r}}_{\mathrm{fin}} = F[0]$. By Corollary 32 we have

$$A^{\mathrm{r}}_{\mathrm{fin}} \xrightarrow{\Psi} F[0]_\lambda \xrightarrow{\Phi} (\mathrm{End}\, M(\lambda))^{\mathrm{r}}_{\mathrm{fin}},$$

and $\Phi\Psi$ is the restriction of the canonical map $A \to \mathrm{End}\, M(\lambda)$. It is known that this restriction is surjective (cf. [11, 14]). Since Φ is an embedding, we see that the following holds:

Theorem 35. *The map Φ defines an isomorphism between right A-module algebras $F[0]_\lambda$ and $(\mathrm{End}\, M(\lambda))^{\mathrm{r}}_{\mathrm{fin}}$.* □

Remark 10. Let G be a Lie group corresponding to $\mathfrak{g}$, and $H \subset G$ a Cartan subgroup. For each λ the right A-module algebra $F[0]_\lambda$ is a "quantization" of a (G, π_0)-homogeneous Poisson structure on G/H, where π_0 is the Poisson Lie group structure on G defined by the standard quasitriangular solution of the classical Yang-Baxter equation for $\mathfrak{g}$.

6 From dynamical to non-dynamical twists

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ its abelian subalgebra. Suppose

$$J : \mathfrak{h}^* \to (U\mathfrak{g} \otimes U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$$

is a quantum dynamical twist.

Assume that there exists a subalgebra $\mathfrak{v} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ as a vector space. Notice that $U\mathfrak{g} = U\mathfrak{v} \oplus U\mathfrak{g} \cdot \mathfrak{h}$. Denote by $J_{\mathfrak{v}}(\lambda)$ the image of $J(\lambda)$ under the projection onto $(U\mathfrak{v} \otimes U\mathfrak{v})[[\hbar]]$ along $((U\mathfrak{g} \cdot \mathfrak{h} \otimes U\mathfrak{g}) \oplus (U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h}))\,[[\hbar]]$.

Theorem 36. *For any $\lambda \in \mathrm{Dom}\, J$ the element $J_{\mathfrak{v}}(\lambda)$ is a quantum twist for $U\mathfrak{v}$, i.e., $J_{\mathfrak{v}}(\lambda)^{12,3} J_{\mathfrak{v}}(\lambda)^{12} = J_{\mathfrak{v}}(\lambda)^{1,23} J_{\mathfrak{v}}(\lambda)^{23}$, and $(\varepsilon \otimes \mathrm{id})(J_{\mathfrak{v}}(\lambda)) = (\mathrm{id} \otimes \varepsilon)(J_{\mathfrak{v}}(\lambda)) = 1$.*

Proof. We have $J(\lambda) = J_{\mathfrak{v}}(\lambda) + J_{\mathfrak{h}}(\lambda)$, where

$$J_{\mathfrak{h}}(\lambda) \in ((U\mathfrak{g} \cdot \mathfrak{h} \otimes U\mathfrak{g}) \oplus (U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h}))\,[[\hbar]].$$

Denote by $A_{\mathfrak{v}}$ the projection of $A \in (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g})[[\hbar]]$ onto $(U\mathfrak{v} \otimes U\mathfrak{v} \otimes U\mathfrak{v})[[\hbar]]$ along

$$((U\mathfrak{g} \cdot \mathfrak{h} \otimes U\mathfrak{g} \otimes U\mathfrak{g}) \oplus (U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h} \otimes U\mathfrak{g}) \oplus (U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h}))\,[[\hbar]].$$

One can calculate directly, using the fact that $U\mathfrak{v}$ (resp. $U\mathfrak{g} \cdot \mathfrak{h}$) is a subalgebra and a coideal (resp. a left ideal and a coideal) in $U\mathfrak{g}$, that

$$(J(\lambda)^{12,3} J(\lambda - \hbar h^{(3)})^{12})_{\mathfrak{v}} = J_{\mathfrak{v}}(\lambda)^{12,3} J_{\mathfrak{v}}(\lambda)^{12} + (J_{\mathfrak{h}}(\lambda)^{12,3} J_{\mathfrak{v}}(\lambda)^{12})_{\mathfrak{v}}$$

and

$$(J(\lambda)^{1,23} J(\lambda)^{23})_{\mathfrak{v}} = J_{\mathfrak{v}}(\lambda)^{1,23} J_{\mathfrak{v}}(\lambda)^{23} + (J_{\mathfrak{h}}(\lambda)^{1,23} J_{\mathfrak{v}}(\lambda)^{23})_{\mathfrak{v}}.$$

Therefore from (5) it follows that

$$\begin{aligned} &J_{\mathfrak{v}}(\lambda)^{12,3} J_{\mathfrak{v}}(\lambda)^{12} + (J_{\mathfrak{h}}(\lambda)^{12,3} J_{\mathfrak{v}}(\lambda)^{12})_{\mathfrak{v}} = \\ &J_{\mathfrak{v}}(\lambda)^{1,23} J_{\mathfrak{v}}(\lambda)^{23} + (J_{\mathfrak{h}}(\lambda)^{1,23} J_{\mathfrak{v}}(\lambda)^{23})_{\mathfrak{v}}. \end{aligned}$$

Let us prove that in fact $(J_{\mathfrak{h}}(\lambda)^{12,3} J_{\mathfrak{v}}(\lambda)^{12})_{\mathfrak{v}} = (J_{\mathfrak{h}}(\lambda)^{1,23} J_{\mathfrak{v}}(\lambda)^{23})_{\mathfrak{v}} = 0$. Let

$$J_{\mathfrak{h}}(\lambda) = \sum_{m \geq 0} J_{\mathfrak{h}}^{(m)}(\lambda) \hbar^m, \quad J_{\mathfrak{v}}(\lambda) = \sum_{n \geq 0} J_{\mathfrak{v}}^{(n)}(\lambda) \hbar^n,$$

where $J_{\mathfrak{h}}^{(m)}(\lambda) \in (U\mathfrak{g} \cdot \mathfrak{h} \otimes U\mathfrak{g}) \oplus (U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h})$, $J_{\mathfrak{v}}^{(n)}(\lambda) \in U\mathfrak{v} \otimes U\mathfrak{v}$.

Clearly, it is enough to show that for any m, n we have

$$\left(J_{\mathfrak{h}}^{(m)}(\lambda)^{12,3} J_{\mathfrak{v}}^{(n)}(\lambda)^{12}\right)_{\mathfrak{v}} = \left(J_{\mathfrak{h}}^{(m)}(\lambda)^{1,23} J_{\mathfrak{v}}^{(n)}(\lambda)^{23}\right)_{\mathfrak{v}} = 0.$$

Indeed, write

$$J_{\mathfrak{v}}^{(n)}(\lambda) = \sum_i x_i \otimes y_i,$$

where $x_i, y_i \in U\mathfrak{v}$, and

$$J_{\mathfrak{h}}^{(m)}(\lambda) = \sum_j (a_j h'_j \otimes b_j + c_j \otimes d_j h''_j),$$

where $a_j, b_j, c_j, d_j \in U\mathfrak{g}$, $h'_j, h''_j \in \mathfrak{h}$. We have

$$J_{\mathfrak{h}}^{(m)}(\lambda)^{12,3} = (\Delta \otimes \mathrm{id})(J_{\mathfrak{h}}^{(m)}(\lambda)) =$$
$$\sum_j \big((\Delta(a_j)(h'_j \otimes 1 + 1 \otimes h'_j)) \otimes b_j + \Delta(c_j) \otimes d_j h''_j\big)$$

and

$$J_{\mathfrak{v}}^{(n)}(\lambda)^{12} = J_{\mathfrak{v}}^{(n)}(\lambda) \otimes 1 = \sum_i x_i \otimes y_i \otimes 1.$$

Therefore

$$(J_{\mathfrak{h}}^{(m)}(\lambda)^{12,3} J_{\mathfrak{v}}^{(n)}(\lambda)^{12})_{\mathfrak{v}} =$$
$$\left(\sum_{i,j} \big((\Delta(a_j)(h'_j \otimes 1 + 1 \otimes h'_j)) \otimes b_j\big)(x_i \otimes y_i \otimes 1)\right)_{\mathfrak{v}} =$$
$$\left(\sum_{i,j} \big(\Delta(a_j)(h'_j x_i \otimes y_i + x_i \otimes h'_j y_i)\big) \otimes b_j\right)_{\mathfrak{v}} =$$
$$\left(\sum_{i,j} \big(\Delta(a_j)([h'_j, x_i] \otimes y_i + x_i \otimes [h'_j, y_i])\big) \otimes b_j\right)_{\mathfrak{v}} =$$
$$\left(\sum_{i,j} \big(\Delta(a_j)([h'_j, x_i]_{\mathfrak{v}} \otimes y_i + x_i \otimes [h'_j, y_i]_{\mathfrak{v}})\big) \otimes b_j\right)_{\mathfrak{v}} =$$
$$\left(\sum_j \Delta(a_j) \otimes b_j \cdot \left(\left(\sum_i \big([h'_j, x_i]_{\mathfrak{v}} \otimes y_i + x_i \otimes [h'_j, y_i]_{\mathfrak{v}}\big)\right) \otimes 1\right)\right)_{\mathfrak{v}};$$

here $[h'_j, x_i]_{\mathfrak{v}}$ means the projection of $[h'_j, x_i]$ onto $U\mathfrak{v}$ along $U\mathfrak{g} \cdot \mathfrak{h}$, etc.

Now recall that $\mathrm{ad}_h(J(\lambda)) = 0$ for all $h \in \mathfrak{h}$. Projecting this equation onto $(U\mathfrak{v} \otimes U\mathfrak{v})[[\hbar]]$ along $((U\mathfrak{g} \cdot \mathfrak{h} \otimes U\mathfrak{g}) \oplus (U\mathfrak{g} \otimes U\mathfrak{g} \cdot \mathfrak{h}))[[\hbar]]$, we get

$$\sum_i ([h, x_i]_{\mathfrak{v}} \otimes y_i + x_i \otimes [h, y_i]_{\mathfrak{v}}) = 0$$

for all $h \in \mathfrak{h}$. Combining this with the previous computation, we see that $(J_{\mathfrak{h}}^{(m)}(\lambda)^{12,3} J_{\mathfrak{v}}^{(n)}(\lambda)^{12})_{\mathfrak{v}} = 0$.

Similarly, $(J_{\mathfrak{h}}^{(m)}(\lambda)^{1,23} J_{\mathfrak{v}}^{(n)}(\lambda)^{23})_{\mathfrak{v}} = 0$.

Finally, the counit condition on $J_{\mathfrak{v}}(\lambda)$ follows easily from (6). □

Example 2. Suppose $\mathfrak{g} = \mathfrak{sl}(2)$. Let x, y, h be the standard basis in $\mathfrak{g}$, and $\mathfrak{h} = \mathbb{C}h$. Notice that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$, where

$$\mathfrak{v} = g \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} g^{-1}, \quad g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Consider the ABRR quantum dynamical twist J for $(\mathfrak{g}, \mathfrak{h})$ (see Example 1). We have $y = b + c$, $x = a - c$, where $b = y + \frac{1}{2}h = g\left(\frac{1}{2}h\right)g^{-1} \in \mathfrak{v}$, $a = x - \frac{1}{2}h = g\left(x + \frac{1}{2}h\right)g^{-1} \in \mathfrak{v}$, $c = -\frac{1}{2}h \in \mathfrak{h}$. Obviously, $[c, b] = b + c$ and $[-c, a] = a - c$.

Lemma 37. *The projection of $y^n = (b + c)^n$ onto $U\mathfrak{v}$ along $U\mathfrak{g} \cdot \mathfrak{h}$ equals $b(b+1)\dots(b+n-1)$.*

Proof. One can easily verify by induction that

$$cb^n = b\left((b+1)^n - b^n\right) + (b+1)^n c.$$

Therefore

$$cf(b) = b\left(f(b+1) - f(b)\right) + f(b+1)c$$

for any polynomial f. Finally,

$$\begin{aligned}
&(b + c) \cdot b(b+1)\dots(b+n-1) = \\
&b^2(b+1)\dots(b+n-1) + \\
&b\left((b+1)(b+2)\dots(b+n) - b(b+1)\dots(b+n-1)\right) + \\
&(b+1)(b+2)\dots(b+n)c = \\
&b(b+1)(b+2)\dots(b+n) + (b+1)(b+2)\dots(b+n)c.
\end{aligned}$$

□

Applying the lemma, we see that

$$J_{\mathfrak{v}}(\lambda) = 1 + \sum_{n \geq 1} \frac{(-1)^n \hbar^n v_n}{n!\lambda(\lambda - \hbar)\dots(\lambda - (n-1)\hbar)},$$

where

$$v_n = b(b+1)\dots(b+n-1) \otimes a(a+1)\dots(a+n-1).$$

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